Course 7: Dynamical Systems

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Introducing Dynamical Systems

Before we get further into the justifications of an ESS as a behaviour we would expect to observe in the real world we need to introduce the idea of a dynamical system. A dynamical system is a system of quantities that changes in time. The population sizes of various plants and animals in an ecosystem, the positions and velocities of celestial bodies, and the concentrations of reagents and reactants in a chemical reaction, are three classic examples of phenomena that can be described by dynamical systems. The reason we are interested in dynamical systems is that we will use them to analyze the way the proportions of various strategies within a population change through time.

An important thing to keep in mind when thinking about dynamical systems is that they are deterministically causal. This means that a given initial condition determines the state of the system at every future time. A dynamical system can be thought of as continuous in which case the system is modeled by an Ordinary Differential Equation (ODE), or it can be thought of as discrete in which case it is appropriate to use a difference equation to describe the dynamics of the system. In this class we will restrict ourselves to continuous dynamical systems and we will use the terms dynamical system and differential equation interchangeably.

When people talk about solving a differential equation they mean finding an explicit formula for $x(t)$, when the only information given is the relationship $\frac{d}{dt} x(t) = f(x(t))$ and an initial condition $x(t_0) = x_0$. An initial condition just specifies where in the space of all possible states the dynamical system starts. The differential equation then dictates how the quantity of interest changes in time. The explicit function $x(t)$ is often called a trajectory of the system. Later we will see how different initial conditions can produce different trajectories. In this class though we are not so interested in the explicit solutions of differential equations what is of more interest to us are the fixed points or

<table>
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<th>Definition 7.1</th>
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<tr>
<td>$\frac{d}{dt} x(t) = f(x(t))$</td>
<td>is a differential equation. It gives the rate of change of some quantity $x$ as a function of $x$ itself. Note that $x$ is itself a function of time.</td>
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equilibria of a dynamical system, and the stability of these fixed points. The notion of a trajectory will still be important for defining stability. We now define a fixed point, and stability.

**Definition 7.2 Fixed Point**

Sometimes called an equilibrium point, a fixed point, \( x^* \), of the system

\[
\frac{d}{dt} x(t) = f(x(t)),
\]

is a point such that

\[
\frac{dx}{dt} \bigg|_{x=x^*} = f(x^*) = 0.
\]

Intuitively \( x^* \) is a point where the rate of change of \( x \) is zero so, if a system is at this point it will not change. This is exactly what it means to be at equilibrium, not changing.

**Definition 7.3 Stability of a Fixed Point**

We say a fixed point is either stable or unstable depending on the behavior of the trajectories in a neighbourhood of the fixed point. If all these trajectories remain near the fixed point, then the point is considered stable, and if any of these trajectories do not remain in a neighborhood of the fixed point, the fixed point is considered unstable. We will not get into the formal definition of neighborhood, but you can think of it as an area “as small as it needs to be” around a given point.

Figuring out the explicit trajectories of the system, even in a small neighbourhood around a fixed point can be extremely difficult. For this reason a useful theorem was proved about how in a small neighbourhood around a fixed point, a linear approximation of the trajectories at the fixed point gives a qualitatively equivalent behaviour. So in essence the real dynamical system may be doing something tricky and intractable, but the stability of a fixed point can be determined using a linear approximation of the dynamical system.

**Exponential Growth Example**

Here is a classic dynamical system. A good model for the size of a bacteria population growing in a Petri-dish (at least initially) is that the rate of growth is proportional to the size of the population. This is precisely what the following differential equation,

\[
\frac{dy}{dt} = ry,
\]

says. The rate of change, \( \frac{dy}{dt} \), of the population is proportional to the size of the population, \( y \). Note that \( r \) is some parameter having to do with growth rate. We assume that \( y \geq 0 \), since a negative population is nonsensical. An initial condition say, \( y(t_0) = y_0 = 5 \) would mean that when we start observing the system at time \( t_0 \), the population is 5 size units big.
The first thing we ask is what are the fixed points of this system. So we set \( \frac{dy}{dt} \bigg|_{y^*} = ry^* = 0 \) and solve for \( y^* \) and we get that the only fixed point is \( y^* = 0 \). In this particular system it is rather easy to find the explicit formula for the trajectories. Given initial condition \( y(0) = y_0 \), the trajectory is \( y(t) = y_0 \cdot e^{rt} \). Using this explicit formula it is clear that if the system is perturbed from the fixed point \( y^* = 0 \) to some small positive \( y' \) then the population will blow up to infinity if \( r > 0 \), that the population will return to \( y^* = 0 \) if \( r < 0 \), and that the system will not change (every point will be a fixed point) if \( r = 0 \). We can then say the following about the stability of the fixed point. If \( r > 0 \) then the fixed point \( y^* = 0 \) is unstable and if \( r < 0 \) then the fixed point \( y^* = 0 \) is stable.

Now we will draw what is called the phase diagram. This is a graph of \( f(x) \) vs. \( x \), with the fixed points of the system labeled, and arrows indicating the direction of the trajectories for the various values of \( x \). It would look like this if \( r \) were positive:

![Phase diagram (positive r)](image)

And it would look like this if \( r \) were negative:

![Phase diagram (negative r)](image)

Notice that the stability of the fixed point is very clear from this picture. If the arrows point to the fixed point then it is stable because the system always moves towards the
fixed point when near the fixed point, and if the arrows point away from the fixed point then the fixed point is unstable because the system moves away from the fixed point when near the fixed point.

**One Dimensional Dynamical Systems**

Finding the stability of the above system’s fixed points was easy because the it was easy to find an explicit formula for the trajectories. This will not often be the case. In general, there will be a system \( \frac{dx}{dt} = f(x) \). We will find it’s fixed points \( x^* \). Then we will evaluate the stability of each fixed point. To do this we will use the linear approximation

\[
\frac{dx}{dt} = f(x^*) + (x - x^*) \cdot f'(x^*). 
\]

Now the differential equation

\[
\frac{dx}{dt} = f(x^*) + (x - x^*) \cdot f'(x^*) 
\]

has the solution

\[
x(t) = e^{f(x^*)t} \cdot (x_0 - x^* + f(x^*)/f'(x^*)) + x^* - f(x^*)/f'(x^*). 
\]

Which is a bit of a headache, but it has the form, \( x(t) = c \cdot e^{f(x^*)t} + b \). From this we can deduce that the fixed point \( x^* \) will be stable if \( f'(x^*) < 0 \) and unstable if \( f'(x^*) > 0 \).

Let’s put all this into a recipe card for one-dimensional dynamical systems.

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**Recipe 7.4 One Dimensional Dynamical Systems**

Given a dynamical system \( \frac{dx}{dt} = f(x) \), do the following:

1. Solve the equation \( f(x^*) = 0 \) for \( x^* \) the fixed point(s) of the dynamical system.
2. Evaluate the stability of the fixed point. If \( f'(x^*) < 0 \) the \( x^* \) is a stable fixed point, if \( f'(x^*) > 0 \) then \( x^* \) is an unstable fixed point, and if \( f'(x^*) = 0 \) we can’t say anything about the stability of the point.
3. Draw the 1-D phase diagram. This is the graph of \( f(x) \) vs. \( x \). Label the fixed points, which are the zero’s of the graph. The zero’s of the graph break the \( x \) axis up into regions. In each region of the \( x \) axis draw an arrow pointing left if \( f(x) < 0 \) in that region and an arrow pointing right if \( f(x) > 0 \) in that region. These arrows show which way the system is moving when it is in that region. (This picture provides insight into the stability conditions of part 2)

Let’s do one slightly harder example before moving on to higher dimension dynamical systems.
**Logistic Growth Example**

Previously we looked at a model where the bacteria population in a Petri-dish grows exponential forever. This predicts that eventually the size of the bacterial colony will be bigger than the Petri-dish, or the lab or even the whole earth, so clearly this exponential growth model is only applicable when the growth of the population is not experiencing any constraints. Another classic dynamical system, called the logistic growth equation takes into account the carrying capacity of the environment that the population is in and gives the following model for growth rate: \( \frac{dx}{dt} = f(x) = r \cdot x \left(1 - \frac{x}{\eta}\right) \), where \( r \) and \( \eta \) are positive parameters. Again growth rate is proportional to the size of the population \( x \) but now it is also proportional to how close the population is to its carrying capacity. We solve \( f(x^*) = r \cdot x^* \left(1 - \frac{x^*}{\eta}\right) = 0 \) for \( x^* \) and get that our fixed points are \( x^* = 0 \) and \( x^* = \eta \). Now \( f'(x) = r \left(1 - \frac{2x}{\eta}\right) = 0 \). To find the stability of the point \( x^* = 0 \) we look at \( f'(0) = r \left(1 - \frac{2 \cdot 0}{\eta}\right) = r > 0 \), and we conclude that the point \( x^* = 0 \) is unstable. To find the stability of the point \( x^* = \eta \) we look at \( f'(\eta) = r \left(1 - \frac{2\eta}{\eta}\right) = -r < 0 \), and we conclude that the point \( x^* = 0 \) is stable.

Now we draw the phase diagram.
Higher Dimension Dynamical Systems

Thus far we have only looked at systems that keep track of one quantity as it changes in time. It is quite possible to study systems of the changes in several related quantities. In this case our dynamical system will be a set of differential equations, one equation for each quantity we want to keep track of. If we had \( n \) quantities we wanted to track, we would need \( n \) equation and our system would look like this:

\[
\begin{align*}
\frac{dx_1}{dt} &= f_1(x_1, x_2, x_3, \ldots, x_n) \\
\frac{dx_2}{dt} &= f_2(x_1, x_2, x_3, \ldots, x_n) \\
\frac{dx_3}{dt} &= f_3(x_1, x_2, x_3, \ldots, x_n) \\
& \vdots \\
\frac{dx_n}{dt} &= f_n(x_1, x_2, x_3, \ldots, x_n)
\end{align*}
\]

So the rate of change of each quantity potentially depends on both itself and every other quantity. Again we are interested in the fixed points of such systems and their stability. We find the fixed points by solving for the point \( \mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_n^*) \) which satisfies the following system of equations:

\[
\begin{align*}
f_1(x_1, x_2, x_3, \ldots, x_n) &= 0 \\
f_2(x_1, x_2, x_3, \ldots, x_n) &= 0 \\
f_3(x_1, x_2, x_3, \ldots, x_n) &= 0 \\
& \vdots \\
f_n(x_1, x_2, x_3, \ldots, x_n) &= 0
\end{align*}
\]

Now we address the problem of finding a fixed points stability. Like the one dimensional systems it is often impossible to find the exact trajectories the system. Fortunately the notion of a linear approximation generalizes to higher dimensional systems. In this course we will deal only with one and two dimensional systems, so the remainder of this section will just be about two dimensional systems. A 2-D system has the form:

\[
\begin{align*}
\frac{dx}{dt} &= f(x, y) \\
\frac{dy}{dt} &= g(x, y)
\end{align*}
\]

We find the fixed point of the system by solving the following system of equations:

\[
\begin{align*}
f(x, y) &= 0 \\
g(x, y) &= 0
\end{align*}
\]

A particularly informative way of solving this system is by finding the isoclines of \( f \) and \( g \).
Recipe 7.5 Finding the Fixed Points In 2-D systems using Isoclines.

Given a function of two variables, say \( f(x,y) \), the isoclines of this function are the curves in the \( xy \)-plane where \( f(x,y) = 0 \). A fixed point of a 2-D system, \( \frac{dx}{dt} = f(x,y) \) and \( \frac{dy}{dt} = g(x,y) \) is the intersection of the isoclines of \( f(x,y) \) and \( g(x,y) \). Note that on the isoclines of \( f(x,y) \) the \( x \) quantity is unchanging, and on the isoclines of \( g(x,y) \) the \( y \) quantity is unchanging. Then on the intersection of the isoclines, both the \( x \) and \( y \) quantities are unchanging, and hence these intersections are the fixed points of the system.

Definition 7.6 The Jacobian of a 2-D systems and Stability Analysis.

Given the following 2-D system,

\[
\begin{align*}
\frac{dx}{dt} &= f(x,y) \\
\frac{dy}{dt} &= g(x,y)
\end{align*}
\]

The Jacobian is a matrix \( J \) defined by

\[
J = \begin{pmatrix}
\frac{\partial f(x,y)}{\partial x} & \frac{\partial f(x,y)}{\partial y} \\
\frac{\partial g(x,y)}{\partial x} & \frac{\partial g(x,y)}{\partial y}
\end{pmatrix}
\]

Given that \((x^*,y^*)\) is a fixed point, that is \( f(x^*,y^*) = 0 \) and \( g(x^*,y^*) = 0 \), we can determine the stability of the fixed point as follows. We consider the matrix:

\[
J|_{x=x^*,y=y^*} = J(x^*,y^*) = \begin{pmatrix}
\frac{\partial f(x^*,y^*)}{\partial x} & \frac{\partial f(x^*,y^*)}{\partial y} \\
\frac{\partial g(x^*,y^*)}{\partial x} & \frac{\partial g(x^*,y^*)}{\partial y}
\end{pmatrix}
\]

Generally one would find the eigenvalues of this matrix. If the largest real part of all the eigenvalues is positive then the fixed point \((x^*,y^*)\) is unstable and if the largest real part of all the eigenvalues is negative then the fixed point \((x^*,y^*)\) is stable.

Once we have the fixed points of a 2-D system we then want to analyze their stability. To do this we use what is called a Jacobian. The definition of the Jacobian matrix and its use in finding stability is as follows.

Calculating eigenvalues is often difficult. In the two dimensional case everything we want to know about the stability of a fixed point can be determined by two values \( T \) and \( D \) which can be calculated from the matrix \( J(x^*,y^*) \), so don’t worry if you are uncomfortable with or have never heard of matrices or eigenvalues. The main thing to keep in mind is that the stability determining process described above is essentially a generalization of the linear approximation technique we used in the 1-D case. We will now show how to calculate the \( T \) and \( D \), and what these values mean in terms of the stability of the fixed point.
Now that we have a way to determine which of six categories a fixed point is in, we show what it means for a fixed point to be a stable node, an unstable node, a stable focus, an unstable focus, a center, or a saddle point. The best way to do this is with the aid of a phase diagram. In a 1-D system the phase diagram was a graph of $f(x)$ vs. $x$. Now we have two quantities that we are tracking so we would need to draw two 3-D graphs one of $z = f(x,y)$ vs. $(x,y)$ and one of $z = g(x,y)$ vs. $(x,y)$, if we wanted to do exactly what we’d done in the 1-D case. Drawing and interpreting 3-D graphs can be difficult, so for a 2-D system we define the phase diagram as follows. It is a picture in the $xy$-plane, note that each point in the $xy$-plane corresponds to a state that the system could potentially be in. In this picture we draw the isoclines of $f(x,y)$ and $g(x,y)$. We label the fixed points, which are the intersections of the isoclines. The isoclines divide the $xy$-plane into regions. In each region we draw little arrows to show which general direction the trajectories go in that region. An alternative way to think of the phase diagram is as the picture made by intersecting our 3-D graphs, $z = f(x,y)$ vs. $(x,y)$ and $z = g(x,y)$ vs. $(x,y)$, with the $xy$-plane, (which can be thought of as $z = 0$).

Here are the phase diagrams of all the different types of fixed points. In each picture: The big dot is the fixed point; The horizontal and vertical lines are the isoclines; The curves with arrows on the end are some possible trajectories; And the little clusters of arrows give the general direction of the trajectories in the corresponding region of the $xy$-plane. The boundaries of these regions are defined by the isoclines.

**Saddle Point**

A saddle point is a fixed point where the trajectories are attracted to the fixed point for some time until they come to close to the fixed point, and they start moving away from the fixed point. Here is a picture of the trajectories around a saddle point.
Stable Node
A stable node is a fixed point where the trajectories are directly attracted to the node. Here is a picture of the trajectories around a stable node.

Unstable Node
An unstable node is a fixed point where the trajectories are directly repulsed from the node. Here is a picture of the trajectories around an unstable node.
Stable Focus
A stable focus is a fixed point where the trajectories are attracted to the focus, but they spiral into it instead of being directly attracted. Here is a picture of a trajectory around a stable focus.

Unstable Focus
An unstable focus is a fixed point where the trajectories are repulsed from the focus, but they spiral away from it instead of being directly repulsed. Here is a picture of a trajectory around an unstable focus.

Center
A center is a fixed point where the trajectories are neither repulsed nor attracted, instead the trajectories just orbit the fixed point from the node. Here is a picture of some trajectories around a center.
Lotka-Volterra Predator-Prey Example

Consider two populations of critters. One of the populations consists of prey, let’s say ferns, and the other population consists predators, let’s say deer. In this simple ecosystem the fern population just grows exponentially when left to its own devices. However the grass population also is eaten up at a rate proportional to the product of the size of the deer population with the fern population. The idea being that deer find ferns randomly and so if there are more deer and more ferns then the chances of a deer finding and eating a fern are higher. The deer population is assumed to grow at some rate proportional to the size of the deer population and the size of the fern population available for consumption. The deer population is also assumed to shrink at some rate proportional to their population size. If we let \( x \) be the size of the fern population and \( y \) be the size of the deer population then the following dynamical system summarizes the above paragraph.

\[
\frac{dx}{dt} = f(x,y) = (r - ay)x \\
\frac{dy}{dt} = g(x,y) = (bx - \mu)y
\]

The parameters \( r, a, b, \) and \( \mu \), which we assume are all positive, can be interpreted as follows. \( r \) has to do with the growth rate of the ferns, \( a \) has to do with the rate at which ferns get eaten (ate) by deer, \( b \) has to do with the birth rate of deer given how many ferns are around, and \( \mu \) has to do with the rate at which deer die. Numbers are often easier to work with than letters so let's set the parameters as follows: \( r = 4, a = 3, b = 2, \mu = 1 \).

The first thing we do is find the isoclines. The isoclines for \( f(x,y) = (r - ay)x \) are obtained by solving \( (r - ay)x = 0 \) for a relationship between \( x \) and \( y \). The isoclines for \( f(x,y) \) are the vertical line \( x = 0 \) and the horizontal line \( y = r/a \). The isoclines for \( g(x,y) = (bx - \mu)y \) are also obtained by solving \( (bx - \mu)y = 0 \) for a relationship between \( x \) and \( y \). The isoclines for \( g(x,y) \) are the horizontal line \( y = 0 \) and the vertical line \( x = \mu/b \). So now we can draw a phase diagram and see where our fixed points are.
So we have one fixed points at (0,0) and (μ/b,r/a). Each of these points is an intersection of an isocline of \( f(x,y) \) and an isocline of \( g(x,y) \). Note that the points (μ/b,0) and (0,r/a) are not fixed points since (0,r/a) is the intersection the two isoclines of \( f(x,y) \), and (μ/b,0) is the intersection of the two isoclines of \( g(x,y) \). The little clusters of arrows are found by noting \( f(x,y) > 0 \) with \( \rightarrow \), \( f(x,y) < 0 \) with \( \leftarrow \), \( g(x,y) > 0 \) with \( \uparrow \), and \( g(x,y) < 0 \) with \( \downarrow \), in each of region of the phase diagram.

We now do a stability analysis on our two fixed points. We compute the Jacobian.

\[
J = \begin{pmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y}
\end{pmatrix} = \begin{pmatrix}
r - ay & -ax \\
b \mu & bx - \mu
\end{pmatrix}
\]

First we analyze the fixed point (0,0). We evaluate the Jacobian at the point (0,0).

\[
J(0,0) = \begin{pmatrix}
r - a \cdot 0 & -a \cdot 0 \\
b \cdot 0 & b \cdot 0 - \mu
\end{pmatrix}
\]

For this fixed point \( D = -r \mu \). We assumed that \( r \) and \( \mu \) were positive so, \( D = -r \mu < 0 \), and we conclude that (0,0) is a saddle point. Next we analyze the fixed point \( (\mu/b,r/a) \). We evaluate the Jacobian at the point \( (\mu/b,r/a) \).

\[
J(\mu/b,r/a) = \begin{pmatrix}
r - a \cdot \frac{br}{a} & -a \mu \\
\frac{br}{a} & b \mu - b \mu - \mu
\end{pmatrix} = \begin{pmatrix}
0 & -a \mu \\
\frac{br}{a} & b
\end{pmatrix}
\]

For this fixed point \( D = 0 - \frac{-a \mu}{b} \cdot \frac{br}{a} = \mu r > 0 \) and \( T = 0 \). We conclude that the point \( (\mu/b,r/a) \) is a center.
Last we check to see if the stability and type of our fixed point agrees with our phase diagram. So we sketch a sample trajectory and see if it agrees with the arrows in our phase diagram and the type and stability of our fixed points.

Everything seems reasonable. We are done analyzing the model mathematically, now what kind of story can we make up about the real world based on this model. First let’s think about what the trajectory we’ve drawn means. In the bottom right quadrant relative to the center fixed point there are more ferns, \( x \), than deer, \( y \), in some sense so both populations are growing. At some point there are too many deer and the fern population begins to shrink from over predation, while the deer population continues to grow, at this point we are in the upper right quadrant relative to the center. Eventually there are not enough ferns to sustain growth in the deer population so the deer population begins to decrease as well, now we are in the upper left quadrant relative to the center. Then the deer population drops low enough that the fern population can begin to regenerate, and we are in the bottom left quadrant. Eventually there are enough ferns that the deer population begins to grow again, and we are back in the bottom right quadrant and the whole cycle begins again. So about the real world, we can say that if there is ever an ecological system roughly like the one we described then we would expect the abundance of prey and predators to oscillate in an endless cycle. This pattern of oscillation has been observed and recorded in many situations. For instance the Hudson’s Bay Company has excellent records of how pelts were collected of various animals, and it turns out that the number of pelts of predatory animals oscillated just as predicted by this model (with some tweaking) with the number of pelts of a given predator’s primary prey. Sweet!
Summary
We summarize by giving a condensed recipe for what should be done with a 2-D dynamical system.

**Recipe 7.7 2-D Dynamical Systems: Phase Plane and Fixed Point Stability Analysis.**

Given a 2-D system,

\[
\frac{dx}{dt} = f(x, y) \\
\frac{dy}{dt} = g(x, y)
\]

1. Find the isoclines and use these to find the fixed points as in recipe 7.5
2. Draw a phase diagram, this a picture of the isoclines and the fixed points in the xy-plane. Draw arrows to indicate the sign of \(f(x, y)\) and \(g(x, y)\) in each of the regions bounded by the isoclines.
3. Calculate the Jacobian of the system

\[
J = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}
\]

4. Evaluate the Jacobian at each fixed point. Then calculate \(D\) and \(T\) to determine the type and stability of each fixed point as in recipe 7.7
5. Check that the stability of each fixed point agrees with the information in your phase diagram. A good way to do this is sketch a few sample trajectories.
6. If the dynamical system has a story related to it, translate the mathematical analysis into a relevant continuation of that story, if not on paper then at least for your self.

The following picture is also a useful condensed version of recipe 7.7.
The only case not included in this picture is when $D$ is positive and $T = 0$ in which case the fixed point is a center. So keep that in mind. In the next section we will put all of these tools to use and discover the relationship between a NE and the equilibria of certain dynamical systems.