Course 8: Replicator Dynamics

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Introduction

In the previous few courses we developed the idea of an ESS, which is a refinement of a NE. We then gave a crash course in dynamical systems. These two ideas merge beautifully in the idea of Replicator Dynamics. As you may have guessed, we will be studying the dynamics of the proportions of strategies present within a population. We will look at two different ways that strategies can spread. We will consider the case where strategies are inherited and so strategies spread proportionally to their success. Where success is measured in terms of fitness. We will also consider the case where the players in the population are sentient actors who observe some of the other strategies being used around them and the relative success of these strategies, and then potentially choose to switch to a different strategy.

Replicator Equation I

Consider a population of clever critters, like humans or monkeys, dolphins, crows, governments etc. Suppose that there is some evolutionary game (two-player and symmetric) that these critters play with each other. This game has a set of pure strategies $S = \{s_1, s_2, ..., s_n\}$, and a payoff function $\pi(s_i, s_j)$ being the payoff to a critter playing strategy s_i against a critter playing s_j. We allow only pure strategies. We denote the proportion of the population playing strategy s_i at time *t* by $p_i(t)$, and call $P(t) = \{p_1(t), p_2(t), ..., p_n(t)\}$ the strategy profile of the population at time *t*. We assume that the critters are equally likely to play the game with any other critter in the population, so the expected payoff to a critter playing strategy s_i in at time *t*, in a population with strategy profile $P(t)$ is
$\pi(s_i, s_1) \cdot p_1(t) + ... + \pi(s_i, s_n) \cdot p_n(t) = \sum_{j=1}^{n} \pi(s_i, s_j) \cdot p_j(t)$, which we denote $\pi_i(t)$. We are interested in how these p_i's change through time. Note that the strategy profile, $P(t)$, is changing in time, so the payoff of a given strategy is a function of time. To analyze how the p_i's change through time, we need to specify a rule of change for the p_i's. This rule of change should be the result of some sensical assumptions about the population. We now derive this rule of change. If you are only interested in what the Replicator Equation is, and how it is used, and not interested in where it comes from you can skip the next few paragraphs.

Suppose in a small time increment, *dt*, each of the critters learns of another randomly chosen critters strategy and it's current payoff with probability $\alpha \cdot dt > 0$. α can be

thought of as the rate at which information spreads in the population. The critter then switches to this new strategy with a probability proportional to how much better this alternative strategy appears. We will denote this probability of switching from strategy s_i to strategy s_j at time *t*, given that a critter playing s_i has learned of strategy s_j, as $p_{ij}(t)$, and define it as follows:

$$p_{ij}(t) = \begin{cases} \beta(\pi_j(t) - \pi_j(t)) & \text{for } \pi_j(t) > \pi_j(t) \\ 0 & \text{for } \pi_j(t) < \pi_j(t) \end{cases}$$

β can be thought of as how willing the critters are to try a new strategy. A high β means that critters eager to change strategies, and a low β means that critters are reluctant to change their strategy. We require that β be positive and small enough to ensure that $0 < \beta(\pi_j(t) - \pi_j(t)) < 1$. So the probability of switching from s_i to s_j in *dt* time units is $\alpha \cdot dt \cdot p_j(t) \cdot \beta(\pi_j - \pi_i)$ if s_j is a better strategy at the moment and zero if s_i is a better strategy at the moment.

We have defined a probabilistic strategy switching process. Given $p_i(t)$ we write the expectation of $p_i(t+dt)$, denoted $\mathbf{E}[p_i(t+dt)]$ as follows:

$$\mathbf{E}[p_i(t+dt)] =$$
$$p_i(t) - p_i(t) \cdot \alpha \cdot dt \cdot \sum_{j=i+1}^{n} p_j(t) \cdot \beta(\pi_j(t) - \pi_i(t)) + \sum_{j=1}^{i} \alpha \cdot dt \cdot p_j(t) \cdot p_i(t) \cdot \beta(\pi_j(t) - \pi_i(t))$$
$$= p_i(t) + p_i(t) \cdot \alpha \cdot dt \cdot \sum_{j=i+1}^{n} p_j(t) \cdot \beta(\pi_i(t) - \pi_j(t)) + \sum_{j=1}^{i} \alpha \cdot dt \cdot p_j(t) \cdot p_i(t) \cdot \beta(\pi_j(t) - \pi_i(t))$$
$$= p_i(t) + p_i(t) \cdot \alpha \cdot dt \cdot \sum_{j=1}^{n} p_j(t) \cdot \beta(\pi_i(t) - \pi_j(t))$$
$$= p_i(t) + p_i(t) \cdot \alpha \cdot dt \cdot \beta(\pi_i(t) - \overline{\pi}(t))$$

Where $\overline{\pi}(t)$ is the average payoff in the population at time *t*.

Where did this expression come from? Note that:

net change in critters playing s_i =
new critters playing s_i – critters playing s_i changing to other strategies

Now

Expected number of new critters playing s_i =
(Expected number of critters that learn a new strategy)$\cdot$(prob they learn s_i)$\cdot$(prob they switch to s_i)
$= (\text{Total number of critters} \cdot \alpha \cdot \text{dt}) \cdot (p_i(t)) \cdot \sum_{j=1}^{n} p_j(t) \cdot p_{ji}(t)$

Where (prob they switch to s_i) is calculated by conditioning on which strategy they were playing in the first place.

And

Expected number of critters playing s_i that switch to other strategies =
(Expected number of critters playing s_i that learn new strategies) $\cdot$ (prob they switch strategies)
= (Total number of critters $\cdot\, p_i(t)\cdot\alpha\cdot dt)\cdot\sum_{j=1}^{n} p_j(t)\cdot p_{ij}(t)$

Where the (prob they switch strategies) is calculated by conditioning on which strategy they learn of.

These word equations are in terms of the numbers of critters, but if we divide both sides of these expressions by the total population size we are left expression exclusively in terms of proportions. We can do this because the strategy switching is happening quickly compared to births and deaths within the population so we can assume that the population size is constant throughout. If we rearrange the indices of the strategies so that $\pi_i(t) < \pi_j(t) \Leftrightarrow i < j$, then we have that $p_{ij}(t) = 0$ for $i < j$ and that $p_{ji}(t) = 0$ for $j < i$, hence the truncated sums in our original expression of $\mathbf{E}[p_i(t+dt)]$.

We assume that the population is large enough that we can replace $\mathbf{E}[p_i(t+dt)]$ by $p_i(t+dt)$. Then subtracting $p_i(t)$ and dividing by dt on both sides of $p_i(t+dt) = p_i(t) + p_i(t)\cdot\alpha\cdot dt\cdot\beta(\pi_i(t)-\bar{\pi}(t))$, we get Replicator Equation I:

$$\frac{p_i(t+dt)-p_i(t)}{dt} = \frac{d}{dt}p_i(t) = \alpha\beta\cdot p_i(t)\cdot(\pi_i(t)-\bar{\pi}(t))$$

Definition 8.1 Replicator Equation I
Given an evolutionary game, in pure strategies $S = \{s_1, s_2, \ldots, s_n\}$, the proportion of the population playing strategy s_i at time *t*, denoted $p_i(t)$, has dynamics described by the differential equation:

$$\frac{d}{dt}p_i(t) = \alpha\beta\cdot p_i(t)\cdot(\pi_i(t)-\bar{\pi}(t))$$

Where α is the rate at which critters learn about other strategies, β is their willingness to change strategies, $\pi_i(t)$ is the expected payoff to a player using strategy s_i at time *t*, and $\bar{\pi}(t)$ is the average payoff in the population at time *t*.

So an evolutionary game produces a dynamical system according to the above rule, if we are in the situation where the critters playing the game are able to learn about other strategies and switch to them. We also need to assume that the time scale on which this strategy learning and switching occurs is much quicker then the ecological time scale of the population, that is the rate of births and deaths within the population. This assumption is so that we can work with a fixed population size.

Let's do an in depth example to see what can be done with this equation. The example will also include some review on ESS, since there is an interesting relationship between the ESS and the Replicator dynamics.

Example: A Population of Hoodlums Playing Chicken

Consider a population of teenage hoodlums who love stealing cars and then playing chicken with each other. Chicken is a game where two hoodlums drive their stolen cars towards each other, as though they were going to have a head on collision. There are two strategies in this game, fearless, and safe. If two fearless hoodlums play against each other they will crash and be physically harmed in a way has a cost h associated with it. If two safe hoodlums play against each other they both swerve away at the same time and loose some esteem from their peers for being “chicken” which has a cost c associated with it. If a fearless hoodlum plays against a safe hoodlum, the safe hoodlum will swerve out of the way, no one will be hurt, the safe hoodlum will loose c for being a chicken, and the fearless hoodlum will win the esteem of her peers of value e. Hoodlums are always curious about what their peers would do and talk to each other about what they would do if they were in a game of chicken. The probability of hearing about another hoodlum’s strategy per small time step dt is $\alpha = 0.3$. Hoodlums are fickle and reckless so they change strategies quite readily, so $\beta = 1/(e+c)$. What is the payoff matrix of this game? Note that it is a little absurd to try and compare the value of the esteem of your peers and physical harm, but supposing that we can the payoff matrix is.

	F	**S**
Fearless	$(-h,-h)$	$(e,-c)$
Safe	$(-c,e)$	$(-c,-c)$

Notice that this is a two-player game symmetric in strategies and payoffs, so it is an evolutionary game. Where *c, h,* and *e* are all positive. We assume that $c < h$ otherwise there are no NE and this is not an interesting example. What are the NE? There are only two pure strategy NE, (Fearless, Safe) and (Safe, Fearless). These are asymmetric so we know that these do not correspond to an ESS. What is the MSNE?
Using the Nash existence theorem and assuming that there is an MSNE of the form $\sigma^* = (p^*)\mathbf{F} + (1-p^*)\mathbf{S}$, where p^* is the NE probability of playing **F**, we have the following equation: $\pi(\mathbf{S},\sigma^*) = \pi(\mathbf{F},\sigma^*)$. Expanding using the expected utility principle gives us: $\pi(\mathbf{S},\mathbf{F})(p^*) + \pi(\mathbf{S},\mathbf{S})(1-p^*) = \pi(\mathbf{F},\mathbf{F})(p^*) + \pi(\mathbf{F},\mathbf{S})(1-p^*)$.
We then plug in the values from our payoff matrix and get,
$-c(p^*) + -c(1-p^*) = -h(p^*) + e(1-p^*)$. Finally we solve this equation for p^* and get that $p^* = \dfrac{(c+e)}{(h+e)}$. We check to make sure that this expression, which is supposed to be a probability, is always between zero and one, which it is given our $c < h$ assumption. We also do an intuitive check. This expression says that the probability of playing **F** should increase if the cost of being a chicken, c, increases or if the potential reward, e, for playing **F** and not crashing increases. It also says that the probability of playing **F** should decrease as the cost of injury, h, increases. So this expression agrees with our qualitative intuition about what how p^* should change with as the various costs and rewards change.
So the ESS of this evolutionary game is $\sigma^* = \left(\dfrac{c+e}{h+e}\right)\mathbf{F} + \left(\dfrac{h-c}{h+e}\right)\mathbf{S}$.

That was all review, now we use Replicator Equation I. According to the equation we have the following dynamical system: $\frac{dp}{dt} = p \cdot \alpha\beta(\pi_{\mathbf{F}} - \bar{\pi})$, where p is the proportion of the population playing strategy **F**. Notice that we could write a similar equation for q the proportion of the population playing **S**, but we don't need to worry about that since $(1-p) = q$, so knowing the dynamics of p gives us the dynamics of q. We want to find fixed points of this system, and their stability. We need to find $\pi_{\mathbf{F}}$ and $\bar{\pi}$. We do this with the expected utility principle to get:

$\pi_{\mathbf{F}} = (p)\pi(\mathbf{F},\mathbf{F}) + (1-p)\pi(\mathbf{F},\mathbf{S}) = e - p(h+e)$ and

$\bar{\pi} = (p)\pi_{\mathbf{F}}(t) + (1-p)\pi_{\mathbf{S}}(t) = -p^2(h+e) + pe + (1-p)(-c) = p(e+c) - p^2(h+e) - c$.

Putting those together we get

$$\pi_{\mathbf{F}} - \bar{\pi} = e - p(h+e) - p(e+c) + p^2(h+e) + c = (e+c)(1-p) - (h+e)p(1-p) =$$
$$(1-p)\big((e+c) - (h+e)p\big)$$

So our Replicator equation is

$$\frac{dp}{dt} = \frac{1}{3}\frac{1}{h+e}p(1-p)\big((c+e) - (h+e)p\big) = p(1-p)\frac{1}{3}\left(\frac{c+e}{h+e} - p\right).$$

We set this equal to zero and solve for the fixed points. There are three: $p^* = 0$, $p^* = 1$, $p^* = \frac{c+e}{h+e}$. If you have set up your Replicator equation correctly zero and one should always be fixed points. Often there will be a fixed point of intermediate value, as is the case with $p^* = \frac{c+e}{h+e}$.

To evaluate the stability of the fixed points we find the derivative of $f(p) = \frac{1}{3}p(1-p)\left(\frac{c+e}{h+e} - p\right)$ and evaluate it at the fixed point. The derivative is:

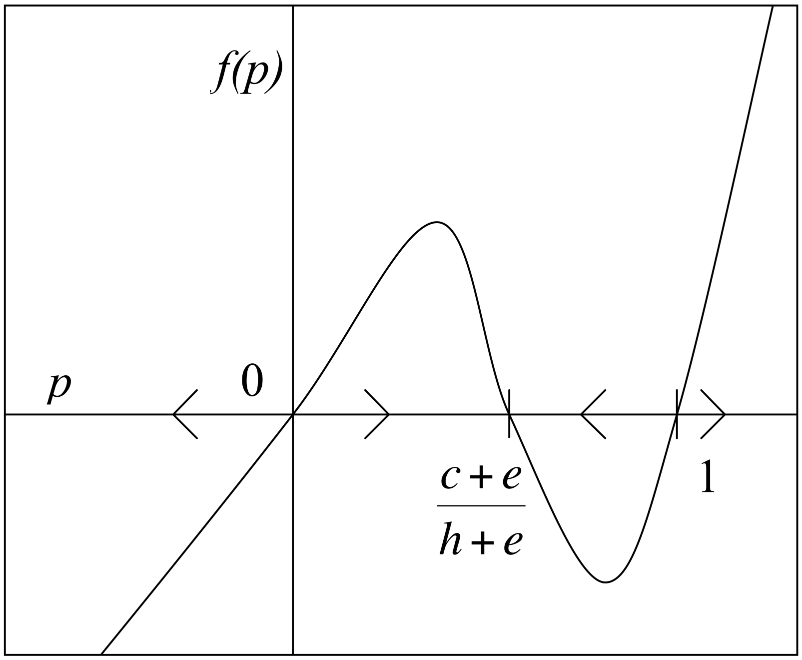
$$f'(p) = \frac{1}{3}(1-2p)\left(\frac{c+e}{h+e} - p\right) + \frac{1}{3}(p-p^2)(-1) = \frac{1}{3}\left(\frac{c+e}{h+e} - 2p\left(1 + \frac{c+e}{h+e}\right) + 3p^2\right).$$

$f'(0) = \frac{1}{3}\left(\frac{c+e}{h+e}\right) > 0$, so $p^* = 0$ is an unstable fixed point.

$f'(1) = \frac{1}{3}\left(-\frac{c+e}{h+e} - 2 + 3\right) > 0$, so $p^* = 1$ is an unstable fixed point.

$f'\left(\frac{c+e}{h+e}\right) = \frac{1}{3}\left(-\frac{c+e}{h+e} + \left(\frac{c+e}{h+e}\right)^2\right) < 0$, so $p^* = \frac{c+e}{h+e}$ is a stable fixed point.

Now to sketch a phase diagram.



There are some interesting observations we would like to make, before we draw conclusions about hoodlums based on our model. The first thing to note is that the ESS proportion and the of hoodlums playing **F** was also the stable equilibrium proportion in the Replicator dynamic. You may be wondering if this is always the case and it is. We state the theorem without proof.

Theorem 8.2 ESS and the Stability of Fixed Points in a Replicator Dynamic
If we have an evolutionary game with payoff function π and strategy set $S = \{s_1, s_2, \ldots, s_n\}$, and ESS $\sigma^* = (p_1^*, p_2^*, \ldots p_n^*)$, where p_i denotes the proportion of the population playing strategy s_i. Then the dynamical system derived from this game using the Replicator equation has a stable fixed point corresponding to the ESS. That is the system:

$$
\begin{aligned}
\frac{dp_1}{dt} &= p_1(\pi_1 - \overline{\pi}) \\
\frac{dp_2}{dt} &= p_2(\pi_2 - \overline{\pi}) \\
&\vdots \\
\frac{dp_n}{dt} &= p_n(\pi_n - \overline{\pi})
\end{aligned}
$$

Has $(p_1^*, p_2^*, \ldots p_n^*)$ as a fixed point.

Note the converse of this theorem is not true, not every stable fixed point of the Replicator dynamic corresponds to an ESS. Also note that the theorem holds even if we throw an α and β, or any other positive constants for that matter, in front of the equations, since multiplying by a positive constants have no effect on the roots or the sign of the derivative and hence no effect on the what fixed points are or their stability. It can be

seen clearly in our example that α and β didn't do anything in term of the fixed points and their stability.

The conclusion that we can draw about teenage hoodlums who steal cars and play chicken is that we expect $\left(\frac{c+e}{h+e}\right)$ of them to play the strategy Fearless, and $\left(\frac{h-c}{h+e}\right)$ to play the strategy Safe, since these are the ESS proportions of the evolutionary game, and it is the stable fixed point of the Replicator dynamic.

Replicator Equation II

Replicator Equation II is essentially the same as Replicator Equation I, except that it is derived from and applied to a different context. Before we considered a constant population of clever critters capable of learning and switching strategies. Now we consider a population of less clever critters, like bacteria, insects, viruses, etc. These critters are only capable of playing a single strategy that they inherited. The idea of inheritance is key to this Replicator Equation.

Definition 8.3 Replicator Equation II
If we have an evolutionary game, in pure strategies $S = \{s_1, s_2, \ldots, s_n\}$, where the payoffs are in terms of fitness, and strategies are inherited and not learned, then the proportion of the population playing strategy s_i at time *t*, denoted $p_i(t)$, has dynamics described by the differential equation:

$$\frac{d}{dt} p_i(t) = p_i(t) \cdot (\pi_i(t) - \overline{\pi}(t))$$

Where $\pi_i(t)$ is the expected fitness of critter using strategy s_i at time *t*, and $\overline{\pi}(t)$ is the average fitness in the population at time *t*.

Notice that no assumptions are made here about the size of the total population, and in fact this equation hold regardless of whether the population as a whole is growing or shrinking or staying the same. We now derive this Replicator equation.

Suppose we have a population of critters that play a single evolutionary game with another random critter from the population, once in their lifetime. The game has pure strategies $S = \{s_1, s_2, \ldots, s_n\}$, and the payoffs of the game determine the number of offspring a critter has. That is $\pi(s_i, s_j)$ is the number of offspring a critter using s_i has when they play with a critter using s_j. Let $y_i(t)$ be the number and $p_i(t)$ be the proportion critters in the population playing strategy s_i at time *t*. Let π_i be the expected number of offspring of critter using strategy s_i and $\overline{\pi}(t)$ be the average number of offspring of all critters in the population. Note that the total population size is $y = \sum_{i=1}^{n} y_i$ and that $p_i = y_i / y$.

By conditioning of what type of critter they play with we calculate that a critter playing s_i has $\sum_{j=1}^{n} p_j \pi(s_i, s_j)$ offspring on average. So if the total number of critters playing s_i in the next generation, given that there are y_i in the current generation, is $y_i \cdot \sum_{j=1}^{n} p_j \pi(s_i, s_j)$, assuming that generation times are infinitesimally small we get the differential equation, $\frac{d}{dt} y_i(t) = y_i(t) \cdot \sum_{j=1}^{n} p_j \pi(s_i, s_j)$.

We have the identity $p_i = y_i / y$, taking the ln of both sides we get $\ln(p_i) = \ln(y_i) - \ln(y)$, we can then differentiate both sides to get, $\frac{1}{p_i}\frac{dp_i}{dt} = \frac{1}{y_i}\frac{dy_i}{dt} - \frac{1}{y}\sum_{j=1}^{n} \frac{dy_j}{dt}$. Substituting in our first differential equation into the first term and using the identity $p_i / y_i = 1/y$ in the second term gives us, $\frac{dp_i}{dt} = p_i\left(\sum_{j=1}^{n} p_j \pi(s_i, s_j) - \sum_{j=1}^{n} \frac{dy_j}{dt}\frac{p_j}{y_j}\right)$.
then substituting or first differential equation into the second term gives us,
$\frac{dp_i}{dt} = p_i\left(\sum_{j=1}^{n} p_j \pi(s_i, s_j) - \sum_{j=1}^{n} p_j \sum_{k=1}^{n} p_k \pi(s_j, s_k)\right)$ which is $\frac{dp_i}{dt} = p_i(\pi_i - \bar{\pi})$.

We use Replicator Equation II just like Replicator Equation I except that there are no constants having to do with rates of learning and willingness to change, and we only use it in the appropriate context. Let's look at an example.

The Birds and The Berries Example

Suppose that in a mountain valley somewhere there is a population of birds and a population of berry bushes. The birds and the berry bushes play a game with each other. The birds have two strategies, and the bushes have two strategies. The strategies of the birds are to either eat berries from the first bush they find, we call this strategy Careless (**L**), or to spend some time searching for a bush with nutritious berries, we call this strategy Safe (**S**). The bushes for their part can either produce tasty and nutritious berries, we call this strategy Tasty (**T**), or produce bland berries and save some energy and nutrients for themselves, we call this strategy Bland (**B**). Now to spend time searching for nutritious berries has a fitness cost *c* compared to a bird that does not search, and eating from a bush with nutritious berries awards some fitness benefit *n* compared to a bird that eats bland berries which can make birds ill and has a cost of *i*. Similarly for a bush, producing nutritious berries has a fitness cost *n*, (same as the benefit to a bird) but having your berries eaten by a bird allows your seeds to be dispersed far and wide which has fitness benefit *d*. The way the game is played is that many times in their lives a bird will land on a bush and both bird and bush will play the strategies that they have inherited, and receive a payoff according to the following matrix.

	T	**B**
S	$(n-s),(d-n)$	$-s,0$
L	$n,(d-n)$	$-i,d$

The question is what proportion of the bird population would we expect to be Safe vs. Lazy and what proportion of the berry population would we expect to be Tasty vs. Bland. We assume that $d > n > i > s$. So we first ask what are the NE and MSNE of this game. It is clear that there are no pure strategy NE, so there must be at least one MSNE by the Nash existence theorem, so let's find it. Assume that there a MSNE (σ^*,τ^*), where $\sigma^* = (p^*)\mathbf{S} + (1-p^*)\mathbf{L}$ and $\tau^* = (q^*)\mathbf{T} + (1-q^*)\mathbf{B}$, where p is the proportion of the bird population that plays Safe and q is the proportion of the berry bush population that plays Tasty. Then using the helpful part of the Nash existence theorem we have the equations: $\pi(\mathbf{S},\tau^*) = \pi(\mathbf{L},\tau^*)$ and $\pi(\mathbf{T},\sigma^*) = \pi(\mathbf{B},\sigma^*)$. Using the expected utility principle we can expand these equations and solve for p^* and q^*. Doing this we get:

$$(q^*)\pi(\mathbf{S},\mathbf{T}) + (1-q^*)\pi(\mathbf{S},\mathbf{B}) = (q^*)\pi(\mathbf{L},\mathbf{T}) + (1-q^*)(\mathbf{L},\mathbf{B})$$

$$(q^*)(n-s) + (1-q^*)(-s) = (q^*)(n) + (1-q^*)(-i)$$

$$q^* = \frac{i-s}{i}$$

and

$$(p^*)\pi(\mathbf{T},\mathbf{S}) + (1-p^*)\pi(\mathbf{T},\mathbf{L}) = (p^*)\pi(\mathbf{B},\mathbf{S}) + (1-p^*)\pi(\mathbf{B},\mathbf{L})$$

$$(d-n) = (1-p^*)d$$

$$p^* = \frac{n}{d}$$

So our MSNE is (σ^*,τ^*) with $\sigma^* = (\frac{n}{d})\mathbf{S} + (\frac{d-n}{d})\mathbf{L}$ and $\tau^* = (\frac{i-s}{i})\mathbf{T} + (\frac{s}{i})\mathbf{B}$.

Now we use Replicator Equation II to come up with a dynamical system that models this pair of populations. Our system is

$$\frac{dp}{dt} = f(p,q) = p \cdot (\pi_{\mathbf{S}} - \overline{\pi})$$

$$\frac{dq}{dt} = g(p,q) = q \cdot (\pi_{\mathbf{T}} - \overline{\pi})$$

Note that the $\overline{\pi}$ in the first equation refers to the average payoff in the bird population but that $\overline{\pi}$ in the second equation refers to the average payoff in the berry bush population.

Now $\pi_{\mathbf{S}} = (q)\pi(\mathbf{S},\mathbf{T}) + (1-q)\pi(\mathbf{S},\mathbf{B}) = (q)(n-s) + (1-q)(-s) = qn - s$ and $\pi_{\mathbf{L}} = (q)\pi(\mathbf{L},\mathbf{T}) + (1-q)(\mathbf{L},\mathbf{B}) = q(n) + (1-q)(-i) = q(n+i) - i$ so for the bird population $\overline{\pi} = p\pi_{\mathbf{S}} + (1-p)\pi_{\mathbf{L}} = pqn - ps + (1-p)q(n+i) - (1-p)i = qn + qi - i + p(-s - qi + i)$ so

$$f(p,q) = p \cdot (\pi_{\mathbf{S}} - \overline{\pi})$$

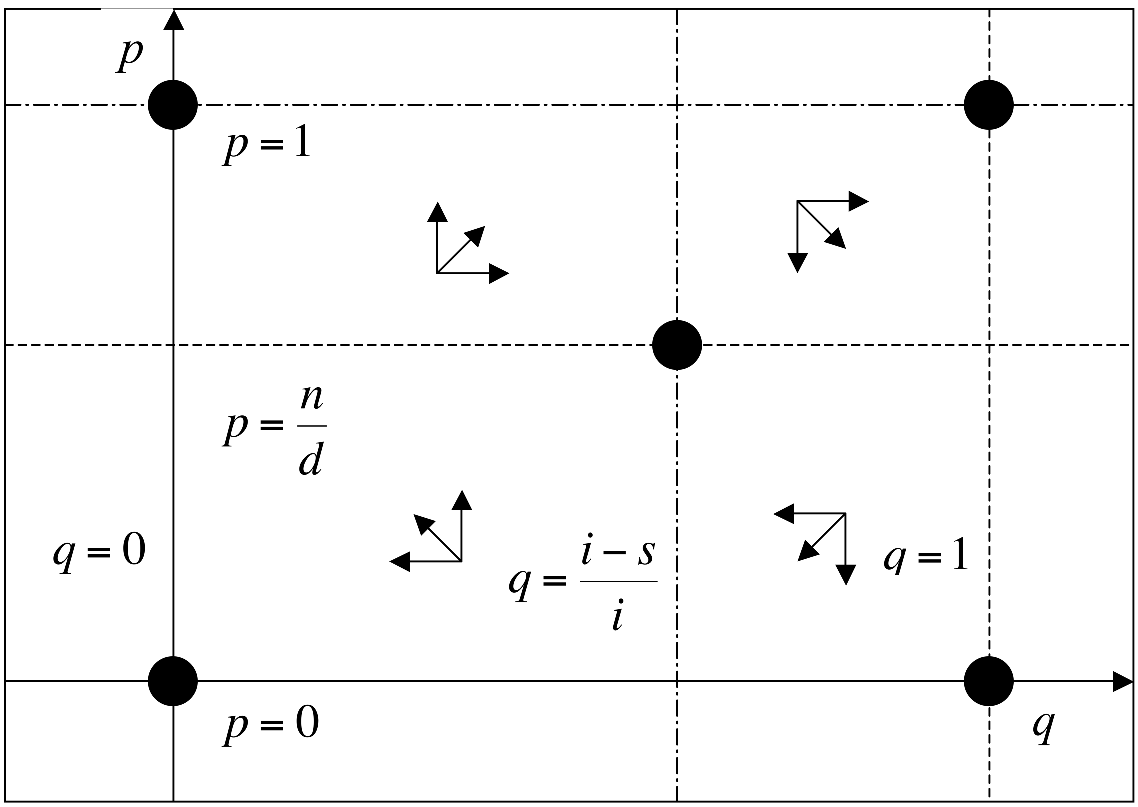
$$= p\big(qn - s - qn - qi + i - p(-s - qi + i)\big)$$

$$= p(1-p)(i - s - qi)$$

For the bushes we have $\pi_{\mathbf{T}} = (p)\pi(\mathbf{T},\mathbf{S}) + (1-p)\pi(\mathbf{T},\mathbf{L}) = d - n$ and $\pi_{\mathbf{B}} = (p)\pi(\mathbf{B},\mathbf{S}) + (1-p)\pi(\mathbf{B},\mathbf{L}) = (1-p)(d)$ so for the bush population $\overline{\pi} = q\pi_{\mathbf{T}} + (1-q)\pi_{\mathbf{B}} = q(d-n) + (1-q)(1-p)d = d - qn + pqd - pd$ so

$$g(p,q) = q \cdot (\pi_{\mathbf{T}} - \bar{\pi}) = q(d - n - d + qn - pqd + pd)$$
$$= q\big(-n(1-q) + pd(1-q)\big)$$
$$= q(1-q)(pd - n)$$

This is a 2-D system so we find the isoclines of $f(p,q)$ and $g(p,q)$, these are $p = 0$, $p = 1$, and $q = \dfrac{i-s}{i}$ for $f(p,q)$, and $q = 0$, $q = 1$, and $p = \dfrac{n}{d}$ for $g(p,q)$. We start drawing our phase diagram and find the intersections of these isoclines, as these are the fixed points of the system. Where the dashed lines are the isoclines of $g(p,q)$ and the dot-dash lines are the isoclines $f(p,q)$, and the big black dots where they intersect are the fixed points of the system.



So there are five fixed points $(0,0),(0,1),(1,0),(1,1)$ and $(n/d,(i-s)/i)$
To access the stability of these points we find the Jacobian of the system.

$$\mathbf{J} = \begin{pmatrix} \dfrac{\partial f}{\partial p} & \dfrac{\partial f}{\partial q} \\ \dfrac{\partial g}{\partial p} & \dfrac{\partial g}{\partial q} \end{pmatrix} = \begin{pmatrix} (1-2p)(i-s-q) & (p-p^2)(-i) \\ (1-2q)(pd-n) & (q-q^2)(d) \end{pmatrix}$$

Now we evaluate the Trace **T**, not to be confused with the strategy tasty, and the determinant **D**, at each of the fixed points. Notice that for the points $(0,0),(0,1),(1,0)$, and $(1,1)$ the expressions $(q-q^2)$ and $(p-p^2)$ are always zero and hence the determinant is always zero for these points, so we can't say anything about their

stability, but we can guess from drawing trajectory arrows that they are all saddle points. Now

$$\mathbf{J}(\frac{n}{d},\frac{i-s}{i}) = \begin{pmatrix} (1-2\frac{n}{d})(i-s-\frac{i-s}{i}) & (\frac{n}{d}-\frac{n^2}{d^2})(-i) \\ (1-2\frac{i-s}{i})(\frac{n}{d}d-n) & (\frac{i-s}{i}-\frac{(i-s)^2}{i^2})(d) \end{pmatrix} = \begin{pmatrix} 0 & (\frac{n}{d}-\frac{n^2}{d^2})(-i) \\ 0 & (\frac{i-s}{i}-\frac{(i-s)^2}{i^2})(d) \end{pmatrix}$$

So $\mathbf{D}=0$ and $\mathbf{T}=\left(\frac{i-s}{i}\right)\left(\frac{s}{i}\right)d>0$, now again $\mathbf{D}=0$ and we can't really say anything about the stability, but using the theorem about how an ESS correspond to an asymptotically stable fixed point, and the trajectory arrows we can guess that this is a stable focus. The conclusion that we draw is that n/d of the birds will play Safe and $(i-s)/i$ of the bushes will play Tasty.

Summary

Given an evolutionary game it is possible to derive a dynamical system using the appropriate replicator equation, often called the replicator dynamic. We use Replicator Equation I when strategies are learned, and Replicator Equation II when strategies are inherited. There is an interesting relationship between the ESS of the game and the fixed points of the replicator dynamic, which is that the ESS corresponds to an asymptotically stable fixed point of the replicator dynamic.