How Long to Wait Example

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The players are two neo-luddites, Byron and Ludd, trying to meet eachother at time 0. We will assume that each one arrives with a uniform distribution between the times -1 and 1. The players' strategies are how long the choose to wait for their friend before giving up and leaving, we will denote these t_b and t_l, and note that wait times must always be positive, and should never be longer than two time units. And we will denote the arrival times of Byron and Ludd a_b and a_l respectively.

The payoff to a neo-luddite is determined in the following way. They lose happiness points linearly as a function of time spent waiting and they recieve a lump sum of happiness points if they do actually mangae to meet their friend. Let m be the amount of happiness they gain from meeting and let ct the happines they lose if they wait t time units.

First we calculate Byron's payoff function. There are four distinct cases that can occur.

1. Byron can arrive first wait his entire wait time, t_b and not have Ludd show up which gives Byron a payoff of $-ct_b$ and happens when $a_b + t_b < a_l$.
2. Byron can arrive first and have Ludd show up before Byron gives up on him Gives Byron a payoff of $m - c(a_l - a_b)$ and happens when $a_b < a_l < a_b + t_b$.
3. Byron can arrive second and have Ludd be there waiting for him which gives Byron a payoff of m and happens when $a_l < a_b < a_l + t_l$.
4. Byron can arrive second and Ludd can have given up waiting and left already which gives Byron a payoff of $-ct_b$ and happens when $a_l + t_l < a_b$

Now we need to calculate the probability of each case occuring.

1. The probability of case 1 is:

$$P(a_b + t_b < a_l) = \int_{-1}^{1-t_b} \frac{1}{2} \int_{a_b+t_b}^{1} \frac{1}{2} \, \mathrm{d}a_l \, \mathrm{d}a_b \tag{1}$$

$$= \frac{1}{4} \int_{-1}^{1-t_b} 1 - a_b - t_b \, \mathrm{d}a_b \tag{2}$$

$$= \frac{t_b^2 - 4t_b + 4}{8} \tag{3}$$

2. The probability of case 2 is:

$$P(a_b < a_l < a_b + t_b) = \int_{-1}^{1-t_b} \frac{1}{2} \int_{a_b}^{a_b+t_b} \frac{1}{2}\, \mathrm{d}a_l\, \mathrm{d}a_b + \int_{1-t_b}^{1} \frac{1}{2} \int_{a_b}^{1} \frac{1}{2}\, \mathrm{d}a_l\, \mathrm{d}a_b \tag{4}$$

$$= \frac{1}{4}\left(\int_{-1}^{1-t_b} t_b\, \mathrm{d}a_b + \int_{1-t_b}^{1} 1 - a_b\, \mathrm{d}a_b\right) \tag{5}$$

$$= \frac{1}{4}\left(t_b(2 - t_b) + \frac{t_b^2}{2}\right) \tag{6}$$

$$= \frac{4t_b - t_b^2}{8} \tag{7}$$

3. The probability of case 3 is the same as the probability of case 2 (by the symmetry of the problem) except with t_b replaced by t_l:

$$P(a_l < a_b < a_l + t_l) = \frac{4t_l - t_l^2}{8} \tag{8}$$

4. The probability of case 4 is the same as the probability of case 1 (by symmetry of the problem) except with t_b replaced by t_1:

$$P(a_l + t_l < a_b) = \frac{t_l^2 - 4t_l + 4}{8} \tag{9}$$

As a quick check the probability of all cases should add up to one and that is certainly the case here. Now we are ready to write out Byron's expected payoff function

$$\Pi_b(t_b, t_l) = (-ct_b)\left(\frac{t_b^2 - 4t_b + 4}{8} + \frac{t_l^2 - 4t_l + 4}{8}\right) + (m - c(a_l - a_b))\left(\frac{4t_b - t_b^2}{8}\right) + m\left(\frac{4t_l - t_l^2}{8}\right) \tag{10}$$

$$= m\left(\frac{4(t_b + t_l) - t_b^2 - t_l^2}{8}\right) - c\left(t_b\left(1 + \frac{t_b^2 + t_l^2 - 4(t_b + t_l)}{8}\right) + (a_l - a_b)\left(\frac{4t_b - t_b^2}{8}\right)\right) \tag{11}$$

$$= m\left(\frac{4(t_b + t_l) - t_b^2 - t_l^2}{8}\right) - c\left(t_b + \frac{t_b^3 + t_b t_l^2 - 4(t_b^2 + t_b t_l)}{8} + (a_l - a_b)\left(\frac{4t_b - t_b^2}{8}\right)\right) \tag{12}$$

There is one tricky thing here we have the term $a_l - a_b$ showing up and that is the difference of two random variables, so we need to take their expectation... but we need to be careful the $a_l - a_b$ comes from case 2 where $a_b < a_l < a_b + t_b$, so we need to compute the expectation of $a_l - a_b$ conditional on $a_b < a_l < a_b + t_b$. What we are trying to compute here is on average how much bigger is a_l than a_b given that a_l comes after a_b but before $a_b + t_b$. Because we are working with a uniform distribution it is pretty clear that this will be $\frac{t_b}{2}$. So finally we have:

$$\Pi_b(t_b, t_l) = m\left(\frac{4(t_b + t_l) - t_b^2 - t_l^2}{8}\right) - c\left(t_b + \frac{t_b^3 + t_b t_l^2 - 4(t_b^2 + t_b t_l)}{8} + \left(\frac{4t_b^2 - t_b^3}{16}\right)\right) \tag{13}$$

$$= m\left(\frac{4(t_b + t_l) - t_b^2 - t_l^2}{8}\right) - c\left(t_b + \frac{t_b^3 + 2t_b t_l^2 - 4(t_b^2 + 2t_b t_l)}{16}\right) \tag{14}$$

Now we want to find which t_b will maximize the payoff function for a given t_l to do this we first take the partial derivative of the payoff function with respect to t_b

$$\frac{\partial}{\partial t_b}\Pi_b(t_b,t_l) = m(\frac{1}{2} - \frac{t_b}{4}) - c(1 + \frac{3t_b^2}{16} + \frac{t_l^2}{8} - \frac{t_b}{2} - \frac{t_l}{2}) \tag{15}$$

Now we take the second partial to determine the concavity of the payoff function

$$\frac{\partial^2}{\partial {t_b}^2}\Pi_b(t_b,t_l) = \frac{-m}{4} - c(\frac{3t_b}{8} - \frac{1}{2}) \tag{16}$$

Recall that t_b is constrained to be between 0 and 2. If t_b is greater than $\frac{4}{3}$ the payoff function is certainly concave down, for values of t_b less than $\frac{4}{3}$ the concavity of the function depends on the ratio of c to m. From this we can deduce that depending on the ration of c to m that is the relative cost of waiting versus the relative merits of meeting with a friend, the optimal wait time may or may not be none zero... and further that there will be some critical ratio between c and m where the optimal wait time will switch from 0 to positive

Now to find the critical points of the payoff function we set it's first derivative equal to zero and solve for t_b. Before we do this we should note that this game is perfectly symmetric and so we can deduce that at the Nash equilibrium both players will be playing the same strategy. So to find the Nash equilibrium it is sufficient to set $t_b = t_l = t$ and solve for t when the partial derivative of the payoff function is zero.

$$\left.\frac{\partial}{\partial t_b}\Pi_b(t_b,t_l)\right|_{t_b=t_l=t} = 0 \tag{17}$$

$$m(\frac{1}{2} - \frac{t}{4}) - c(1 + \frac{3t^2}{16} + \frac{t^2}{8} - \frac{t}{2} - \frac{t}{2}) = 0 \tag{18}$$

$$\frac{m}{2} - c + t(c - \frac{m}{4}) - t^2\frac{5c}{16} = 0 \tag{19}$$

Then using the quadratic formula we get that

$$t = \frac{\frac{m}{4} - c \pm \sqrt{(c - \frac{m}{4})^2 - 4(\frac{-5c}{16})(\frac{m}{2} - c)}}{\frac{-5c}{8}} \tag{20}$$

$$= \frac{8}{5} - \frac{2m}{5c} \pm \frac{8}{5c}\sqrt{c^2 - \frac{cm}{8} + \frac{m^2}{16} + \frac{5cm}{8} - \frac{5c^2}{4}} \tag{21}$$

$$= \frac{8}{5} - \frac{2m}{5c} \pm \frac{8}{5c}\sqrt{\frac{-c^2}{4} + \frac{cm}{2} + \frac{m^2}{16}} \tag{22}$$

so t has real solutions if the determinant $\frac{-c^2}{4} + \frac{cm}{2} + \frac{m^2}{16}$ is positive. Using the quadratic formula again to solve for c in terms of m (we could solve for m

in terms of c) we see that the determinant is zero when:

$$c = \frac{\frac{-m}{2} \pm \sqrt{(\frac{m}{2})^2 - 4(\frac{-1}{4})(\frac{m^2}{16})}}{2\frac{-1}{4}} \tag{23}$$

$$= m \pm 2\sqrt{\frac{m^2}{4} + \frac{m^2}{16}} \tag{24}$$

$$= m \pm 2\sqrt{\frac{5m^2}{16}} \tag{25}$$

$$= m(1 \pm \frac{\sqrt{5}}{2}) \tag{26}$$

Then t has real solutions when $m\left(1 - \frac{\sqrt{5}}{2}\right) < c < m\left(1 + \frac{\sqrt{5}}{2}\right)$. but c is positive so it suffices for $0 < \frac{c}{m} < 1 + \frac{\sqrt{5}}{2}$. That we have the optimal wait time given by

$$t = \frac{8}{5} - \frac{2m}{5c} \pm \frac{8}{5c}\sqrt{\frac{-c^2}{4} + \frac{cm}{2} + \frac{m^2}{16}} \tag{27}$$

but which of these two solutions is the sensible one. Intuitively wait time should decrease as cost to waiting increases and increase as reward for meeting the friend increases for this to be the case then

$$t = \frac{8}{5} - \frac{2m}{5c} + \frac{8}{5c}\sqrt{\frac{-c^2}{4} + \frac{cm}{2} + \frac{m^2}{16}} \tag{28}$$

When $\frac{c}{m}$ is greater than $1 + \frac{\sqrt{5}}{2}$ then the cost of waiting is too much and the friends cease to wait for eachother, is perhaps a reasonable interpretation of what is going on.